Sobolev Homeomorphisms and Composition Operators [1]

V. Gol'dshtein and A. Ukhlov

ABSTRACT

We study invertibility of bounded composition operators of Sobolev spaces. The problem is closely connected with the theory of mappings of finite distortion. If a homeomorphism φ of Euclidean domains D and D' generates by the composition rule $\varphi^* f = f \circ \varphi$ a bounded composition operator of Sobolev spaces $\varphi^* : L^1_\infty(D') \to L^1_p(D)$, $p > n-1$, has finite distortion and Luzin N-property then its inverse φ^{-1} generates the bounded composition operator from $L^1_{p'}(D)$, $p' = p/(p-n+1)$, into $L^1_1(D')$.

Introduction

Let φ be a homeomorphism of Euclidean domains $D, D' \subset \mathbb{R}^n$. It is known [1] that φ is a quasiconformal mapping if and only if the composition operator φ^* is an isomorphism of Sobolev spaces $L^1_n(D')$ and $L^1_n(D)$. If φ generates a bounded composition operator of Sobolev spaces $L^1_q(D')$ and $L^1_q(D)$, $q \neq n$, then the inverse homeomorphism φ^{-1} is not necessary generates the bounded composition operator of same spaces. In the more general case homeomorphisms that generate composition operators from $L^1_p(D')$ to $L^1_q(D)$, $1 \leq q \leq p \leq \infty$, are mappings with bounded (p,q)-distortion. These classes of mappings were introduced in [2] as a natural solution of the change of variable problem in Sobolev spaces. Inverse mappings to homeomorphisms with bounded (p,q)-distortion can be described in the same category of mappings with bounded mean distortion. In [3] these classes of mappings were studied in a relation with Sobolev type embedding theorems for non-regular domains.

We recall, that Sobolev space $L^1_p(D)$, $1 \leq p \leq \infty$, consists of locally summable, weakly differentiable functions $f : D \to \mathbb{R}$ with the finite seminorm:

$$\|f \mid L^1_p(D)\| = \||\nabla f| \mid L_p(D)\|, \quad \nabla f = \Big(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\Big).$$

As usually Lebesgue space $L_p(D)$, $1 \leq p \leq \infty$, is the space of locally summable functions with the finite norm:

$$\|f \mid L_p(D)\| = \bigg(\int\limits_D |f|^p \, dx \bigg)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$\|f \mid L_\infty(D)\| = \operatorname*{ess\,sup}_{x \in D} |f(x)|, \quad p = \infty.$$

A mapping $\varphi : D \to \mathbb{R}^n$ belongs to $L^1_p(D)$, $1 \leq p \leq \infty$, if its coordinate functions φ_j belong to $L^1_p(D)$, $j = 1, \ldots, n$. In this case formal Jacobi matrix $D\varphi(x) = \big(\frac{\partial \varphi_i}{\partial x_j}(x)\big)$, $i, j = 1, \ldots, n$, and its determinant (Jacobian) $J(x,\varphi) = \det D\varphi(x)$ are well defined at almost all points $x \in D$. The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the

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corresponding linear operator $D\varphi(x):\mathbb{R}^n\to\mathbb{R}^n$ defined by the matrix $D\varphi(x)$. We will use the same notation for this matrix and the corresponding linear operator.

Recall that a mapping $\varphi: D\to D'$ is called a *mapping with bounded* (p,q)*-distortion* $1\leq q\leq p\leq\infty$, if φ belongs to Sobolev space $W^1_{1,\mathrm{loc}}(D)$ and the local p-distortion

$$K_p(x)=\inf\{k: |D\varphi|(x)\leq k|J(x,\varphi)|^{\frac{1}{p}},\ x\in D\}$$

belongs to Lebesgue space $L_r(D)$, where $1/r=1/q-1/p$ (if $p=q$ then $r=\infty$).

Mappings with bounded (p,q)-distortion have a finite distortion, i. e. $D\varphi(x)=0$ for almost all points x that belongs to set $Z=\{x\in D: J(x,\varphi)=0\}$.

Necessity of studying of Sobolev mappings with integrable distortion arises in problems of the non-linear elasticity theory [4, 5]. In these works J. M. Ball introduced classes of mappings, defined on bounded domains $D\in\mathbb{R}^n$:

$$A^+_{p,q}(D)=\{\varphi\in W^1_p(D):\operatorname{adj} D\varphi\in L_q(D),\quad J(x,\varphi)>0\quad \text{a. e. in}\quad D\},$$

$p,q>n$, where $\operatorname{adj} D\varphi$ is the formal adjoint matrix to the Jacobi matrix $D\varphi$:

$$\operatorname{adj} D\varphi(x)\cdot D\varphi(x)=\operatorname{Id} J(x,\varphi).$$

The class of mappings with bounded (p,q)-distortion is a natural generalization of mappings with bounded distortion and represents a non-homeomorphic case of so-called (p,q)-quasiconformal mappings [2, 3, 6, 7]. Such classes of mappings have applications to the Sobolev type embedding problems [7–9].

The following assertion demonstrates a connection between Sobolev spaces and mappings with bounded (p,q)-distortion [2]. A homeomorphism φ of Euclidean domains D and D' is a mapping with bounded (p,q)-distortion, $1\leq q\leq p<\infty$, if and only if φ generates a bounded operator of Sobolev spaces

$$\varphi^*: L^1_p(D')\to L^1_q(D)$$

by the composition rule $\varphi^* f=f\circ\varphi$. We call φ^* a composition operator of Sobolev spaces.

In the frameworks of the inverse operator problem in [6] was proved, that if a homeomorphism $\varphi: D\to D'$ generates a bounded composition operator

$$\varphi^*: L^1_p(D')\to L^1_q(D),\ \ n-1<q\leq p<+\infty,$$

then the inverse mapping $\varphi^{-1}: D'\to D$ generates a bounded composition operator

$$(\varphi^{-1})^*: L^1_{q'}(D)\to L^1_{p'}(D'),\ \ q'=q/(q-n+1),\ \ p'=p/(p-n+1).$$

The main result of the article concerns to invertibility of a composition operator in the limit case $p=\infty$.

Theorem A. Let a homeomorphism $\varphi: D\to D'$ has finite distortion, Luzin N-property (the image of a set measure zero is a set measure zero) and generates a bounded composition operator

$$\varphi^*: L^1_\infty(D')\to L^1_q(D),\ \ q>n-1.$$

Then the inverse mapping $\varphi^{-1}: D'\to D$ generates a bounded composition operator

$$(\varphi^{-1})^*: L^1_{q'}(D)\to L^1_1(D'),\ \ q'=q/(q-n+1).$$

The invertibility problem for composition operators in Sobolev spaces is closely connected with a regularity problem for invertible Sobolev mappings. The regularity problem for mappings which are inverse to Sobolev homeomorphisms was studied by many authors. In article [10] was proved that if a mapping $\varphi \in W^1_{n,\mathrm{loc}}(D)$ and $J(x,\varphi) > 0$ for almost all points $x \in D$, then φ^{-1} belongs to $W^1_{1,\mathrm{loc}}(D')$.

The assumption that φ has finite distortion cannot be dropped out. Indeed, consider the function $g(x) = x + u(x)$ on the real line, where u is the standard Cantor function. Let $f = g^{-1}$. Then the derivative $f' = 0$ on the set of positive measure and h^{-1} fails to be absolutely continuous. In this case we can prove only that the inverse homeomorphism has a finite variation on almost all lines [11]. In work [11] was obtained the following result: *if a homeomorphism $\varphi : D \to D'$ belongs to the Sobolev space $L^1_p(D)$, $p > n-1$, then the inverse mapping $\varphi^{-1} : D' \to D$ has a finite variation on almost all lines (belongs to* $\mathrm{BVL}(D')$*).*

In work [12] the local regularity of plane homeomorphisms that belong to Sobolev space $W^1_1(D)$ was studied. For the case of space $\mathbb{R}^n$, $n \geq 3$, recent work [13] contains the following result for domains in $\mathbb{R}^n$, $n \geq 3$: *if the norm of the derivative $|D\varphi|$ belongs to Lorentz space $L^{n-1,1}(D)$ and a mapping $\varphi : D \to D'$ has finite distortion, then the inverse mapping belongs to Sobolev space $W^1_{1,\mathrm{loc}}(D')$ and has finite distortion.* Recall that

$$L^{n-1}(D) \subset L^{n-1,1}(D) \subset \bigcap_{p>n-1} L^p(D).$$

Note, that results about regularity of mappings inverse to Sobolev homeomorphisms follows from Theorem A. Indeed, substituting in the norm inequality for the inverse operator coordinate functions $x_j \in L^1_{p',\mathrm{loc}}(D)$ we see that φ^{-1} belongs to $L^1_{1,\mathrm{loc}}(D')$.

The suggested method of investigation is based on a relation between Sobolev mappings, composition operators of spaces of Lipschitz functions and a change of variable formula for weakly differentiable mappings.

1. Composition operators in Sobolev spaces

A locally integrable function $f : D \to \mathbb{R}$ is *absolutely continuous on a straight line* l having non-empty intersection with D if it is absolutely continuous on an arbitrary segment of this line which is contained in D. A function $f : D \to \mathbb{R}$ belongs to the class $\mathrm{ACL}(D)$ (*absolutely continuous on almost all straight lines*) if it is absolutely continuous on almost all straight lines parallel to any coordinate axis.

Note that f belongs to Sobolev space $L^1_1(D)$ if and only if f is locally integrable and it can be changed by a standard procedure on a set of measure zero (changed to its Lebesgue values at any point where the Lebesgue values exist) so , that a modified function belongs to $\mathrm{ACL}(D)$, and its partial derivatives $\frac{\partial f}{\partial x_i}(x)$, $i = 1, \ldots, n$, exist almost everywhere and are integrable in D. From this point we will use such modified functions only. Note that first weak derivatives of the function f coincide almost everywhere with the usual partial derivatives (see, e.g., [14]).

A mapping $\varphi : D \to \mathbb{R}^n$ belongs to the class $\mathrm{ACL}(D)$, if its coordinate functions φ_j belong to $\mathrm{ACL}(D)$, $j = 1, \ldots, n$.

We will use the notion of approximate differentiability. Let A be a subset of $\mathbb{R}^n$. Density of set A at a point $x \in \mathbb{R}^n$ is the limit

$$\lim_{r\to 0} \frac{|B(x,r)\cap A|}{|B(x,r)|}.$$

Here by symbol $|A|$ we denote Lebesgue measure of the set A.

A linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is called an approximate differential of a mapping $\varphi : D \to \mathbb{R}^n$ at point $a \in D$, if for every $\varepsilon > 0$ the density of the set

$$A_\varepsilon = \{x \in D : |\varphi(x) - \varphi(a) - L(x-a)| < \varepsilon |x-a|\}$$

at point a is equal to one.

A point $y \in \mathbb{R}^n$ is called an approximate limit of a mapping $\varphi : D \to \mathbb{R}^n$ at a point x, if the density of the set $D \setminus \varphi^{-1}(W)$ at this point is equal to zero for every neighborhood W of the point y.

For a mapping $\varphi : D \to \mathbb{R}^n$ we define approximate partial derivatives

$$\operatorname{ap}\frac{\partial \varphi_i}{\partial x_j}(x) = \operatorname{ap}\lim_{t\to 0} \frac{\varphi_i(x+te_j) - \varphi_i(x)}{t}, \quad i,j = 1,...,n.$$

Approximate differentiable mappings are closely connected with Lipschitz mappings. Recall, that a mapping $\varphi : D \to \mathbb{R}^n$ is a Lipschitz mapping, if there exists a constant $K < +\infty$ such that

$$|\varphi(x) - \varphi(y)| \leq K|x-y|$$

for every points $x, y \in D$.

The value

$$\|\varphi \mid \mathrm{Lip}(D)\| = \sup_{x,y\in D} \frac{|\varphi(x) - \varphi(y)|}{|x-y|}$$

we call the norm of φ in the space $\mathrm{Lip}(D)$.

The next assertion describes this connection between approximate differentiable mappings and Lipschitz mappings in details [15].

Theorem 1. Let $\varphi : D \to \mathbb{R}^n$ be a measurable mapping. Then the following assertions are equivalent:
1) The mapping $\varphi : D \to \mathbb{R}^n$ is approximate differentiable almost everywhere in D.
2) The mapping $\varphi : D \to \mathbb{R}^n$ has approximate partial derivatives $\operatorname{ap}\frac{\partial \varphi_i}{\partial x_j}$, $i,j = 1,...,n$ almost everywhere in D.
3) There exists a collection of closed sets $\{A_k\}_{k=1}^{\infty}$, $A_k \subset A_{k+1} \subset D$, such that a restriction $\varphi|_{A_k}$ is a Lipschitz mapping on the set A_k and

$$\left| D \setminus \sum_{k=1}^{\infty} A_k \right| = 0.$$

If a mapping $\varphi : D \to D'$ has approximate partial derivatives $\operatorname{ap}\frac{\partial \varphi_i}{\partial x_j}$ almost everywhere in D, $i,j = 1,\dots,n$, then the formal Jacobi matrix $D\varphi(x) = (\operatorname{ap}\frac{\partial \varphi_i}{\partial x_j}(x))$, $i,j = 1,\dots,n$, and its Jacobian determinant $J(x,\varphi) = \det D\varphi(x)$ are well defined at almost all points of

D. The norm $|D\varphi(x)|$ of the matrix $D\varphi(x)$ is the norm of the linear operator determined by the matrix in Euclidean space $\mathbb{R}^n$.

In the theory of mappings with bounded mean distortion additive set functions play a significant role. Let us recall that a nonnegative mapping Φ defined on open subsets of D is called a *finitely quasiadditive* set function [16] if

1) for any point $x \in D$, there exists δ, $0 < \delta < \operatorname{dist}(x, \partial D)$, such that $0 \leq \Phi(B(x,\delta)) < \infty$ (here and in what follows $B(x,\delta) = \{y \in \mathbb{R}^n : |y - x| < \delta\}$);

2) for any finite collection $U_i \subset U \subset D$, $i = 1, \ldots, k$ of mutually disjoint open sets the following inequality $\sum\limits_{i=1}^{k} \Phi(U_i) \leq \Phi(U)$ takes place.

Obviously, the last inequality can be extended to a countable collection of mutually disjoint open sets from D, so a finitely quasiadditive set function is also *countable quasi-additive.*

If instead of the second condition we suppose that for any finite collection $U_i \subset D$, $i = 1, \ldots, k$ of mutually disjoint open subsets of D the equality

$$\sum_{i=1}^{k} \Phi(U_i) = \Phi(U)$$

takes place, then such set function is said to be *finitely additive.* If the last equality can be extended to a countable collection of mutually disjoint open subsets of D, then such set function is said to be *countable additive.*

A nonnegative mapping Φ defined on open subsets of D is called a *monotone* set function [16] if $\Phi(U_1) \leq \Phi(U_2)$ under the condition, that $U_1 \subset U_2 \subset D$ are open sets.

Note, that a monotone (countable) additive set function is the (countable) quasiadditive set function.

Let us reformulate an auxiliary result from [16] in a convenient for this study way.

Proposition 1. Let a monotone finitely additive set function Φ be defined on open subsets of the domain $D \subset \mathbb{R}^n$. Then for almost all points $x \in D$ the volume derivative

$$\Phi'(x) = \lim_{\delta \to 0, B_\delta \ni x} \frac{\Phi(B_\delta)}{|B_\delta|}$$

is finite and for any open set $U \subset D$, the inequality

$$\int\limits_U \Phi'(x)\, dx \leq \Phi(U)$$

is valid.

A nonnegative finite valued set function Φ defined on a collection of measurable subsets of an open set D is said to be *absolutely continuous* if for every number $\varepsilon > 0$ can be found a number $\delta > 0$ such that $\Phi(A) < \varepsilon$ for any measurable sets $A \subset D$ from the domain of definition of Φ, which satisfies the condition $|A| < \delta$.

Let E be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. Define Lebesgue space $L_p(E)$, $1 \leq p \leq \infty$, as a Banach space of locally summable functions $f : E \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid L_p(E)\| = \biggl(\int\limits_E |f|^p(x)\, dx\biggr)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f \mid L_\infty(E)\| = \operatorname*{ess\,sup}_{x\in E} |f(x)|, \quad p = \infty.$$

A function f belongs to the space $L_{p,\mathrm{loc}}(E)$, $1 \leq p \leq \infty$, if $f \in L_p(F)$ for every compact set $F \subset E$.

For an open subset $D \subset \mathbb{R}^n$ define the seminormed Sobolev space $L^1_p(D)$, $1 \leq p \leq \infty$, as a space of locally summable, weakly differentiable functions $f : D \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(D)\| = \|\nabla f \mid L_p(D)\|, \ \ 1 \leq p \leq \infty.$$

Here ∇f is the weak gradient of the function f, i. e. $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$,

$$\int\limits_D f \frac{\partial \eta}{\partial x_i}\, dx = -\int\limits_D \frac{\partial f}{\partial x_i} \eta\, dx, \quad \forall \eta \in C^\infty_0(D), \quad i = 1, ..., n.$$

As usual $C^\infty_0(D)$ is the space of infinitely smooth functions with a compact support.

Note, that smooth functions are dense in $L^1_p(D)$, $1 \leq p < \infty$ (see, for example [14], [17]). If $p = \infty$ we can assert only that for arbitrary function $f \in L^1_p(D)$ there exists a sequence of smooth functions $\{f_k\}$ converges locally uniformly to f and $\|f_k \mid L^1_\infty(D)\| \to \|f \mid L^1_\infty(D)\|$ (see [17]).

The Sobolev space $W^1_p(D)$, $1 \leq p \leq \infty$, is a Banach space of locally summable, weakly differentiable functions $f : D \to \mathbb{R}$, equipped with the following norm:

$$\|f \mid W^1_p(D)\| = \|f \mid L_p(D)\| + \|f \mid L^1_p(D)\|.$$

A function f belongs to the space $L^1_{p,\mathrm{loc}}(D)$ ($W^1_{p,\mathrm{loc}}(D)$), $1 \leq p \leq \infty$, if $f \in L^1_p(K)$ ($f \in W^1_p(K)$) for every compact subset $K \subset D$. The Sobolev space $\overset{\circ}{L}{}^1_p(D)$ is the closure of the space $C^\infty_0(D)$ in $L^1_p(D)$.

A mapping $\varphi : D \to D'$ belongs to Lebesgue class $L_p(E)$ if its coordinate functions φ_j, $j = 1, \ldots, n$ belong to $L_p(E)$. A mapping $\varphi : D \to D'$ belongs to Sobolev class $W^1_p(D)$ ($L^1_p(D)$) if its coordinate functions φ_j, $j = 1, \ldots, n$, belong to $W^1_p(D)$ ($L^1_p(D)$).

We say that a mapping $\varphi : D \to D'$ generates a bounded composition operator

$$\varphi^* : L^1_p(D') \to L^1_q(D), \ \ 1 \leq q \leq p \leq \infty,$$

if for every function $f \in L^1_p(D')$ the composition $f \circ \varphi \in L^1_q(D)$ and the inequality

$$\|\varphi^* f \mid L^1_q(D)\| \leq K \|f \mid L^1_p(D')\|$$

holds.

Theorem 2. A homeomorphism $\varphi : D \to D'$ between two domains $D, D' \subset \mathbb{R}^n$ generates a bounded composition operator

$$\varphi^* : L^1_\infty(D') \to L^1_q(D), \ \ 1 < q < +\infty,$$

if and only if φ belongs to the Sobolev space $L^1_q(D)$.

Proof. *Necessity.* Substituting in the inequality

$$\|\varphi^* f \mid L^1_q(D)\| \leq K\|f \mid L^1_\infty(D')\|$$

the test functions $f_j(y) = y_j \in L^1_\infty(D')$, $j = 1, ..., n$ we see that φ belongs to $L^1_q(D)$.

Sufficiency. Let a function $f \in L^1_\infty(D') \cap C^\infty(D')$. Then

$$\|\varphi^* f \mid L^1_q(D)\| = \left(\int\limits_D |\nabla(f\circ\varphi)|^q\, dx\right)^{\frac{1}{q}} \leq \left(\int\limits_D |D\varphi|^q |\nabla f|^q(\varphi(x))\, dx\right)^{\frac{1}{q}}$$
$$\leq \left(\int\limits_D |D\varphi)|^q\, dx\right)^{\frac{1}{q}} \|f \mid L^1_\infty(D')\| = \|\varphi \mid L^1_q(D)\| \cdot \|f \mid L^1_\infty(D')\|.$$

For arbitrary function $f \in L^1_\infty(D')$ consider a sequence of smooth functions $f_k \in L^1_\infty(D')$ such that

$$\lim_{k\to\infty} \|f_k \mid L^1_\infty(D')\| = \|f \mid L^1_\infty(D')\|$$

and f_k converges locally uniformly to f in D'. Then, the sequence $\varphi^* f_k$ converges locally uniformly to $\varphi^* f$ in D and is a bounded sequence in $L^1_q(D)$. Since the space $L^1_q(D)$, $1 < q < \infty$, is a reflexive space there exists a subsequence $f_{k_l} \in L^1_q(D)$ which weakly converges to $f \in L^1_q(D)$ and

$$\|\varphi^* f \mid L^1_q(D)\| \leq \liminf_{l\to\infty} \|\varphi^* f_{k_l} \mid L^1_q(D)\|.$$

So, passing to limit when l tends to $+\infty$ in the inequality

$$\|\varphi^* f_{k_l} \mid L^1_q(D)\| \leq K\|f_{k_l} \mid L^1_\infty(D')\|$$

we obtain

$$\|\varphi^* f \mid L^1_q(D)\| \leq K\|f \mid L^1_\infty(D')\|.$$

The next theorem gives a "localization" property of the composition operator on spaces of functions with compact support and/or its closure in L^1_∞.

Theorem 3. Let a homeomorphism $\varphi : D \to D'$ between two domains $D, D' \subset \mathbb{R}^n$ generates a bounded composition operator

$$\varphi^* : L^1_\infty(D') \to L^1_q(D),\ 1 \leq q < +\infty.$$

Then there exists a bounded monotone countable additive function $\Phi(A')$ defined on open bounded subsets of D' such that for every function $f \in \overset{\circ}{L}{}^1_\infty(A')$ the inequality

$$\int\limits_{\varphi^{-1}(A)} |\nabla(f\circ\varphi)|^q\, dx \leq \Phi(A')\operatorname{esssup}_{y\in A'} |\nabla f|^q(y)$$

holds.

PROOF. Let us define $\Phi(A')$ by the following way [2, 6]

$$\Phi(A') = \sup_{f\in \overset{\circ}{L}{}^1_\infty(A')} \left(\frac{\|\varphi^* f \mid L^1_q(D)\|}{\|f \mid \overset{\circ}{L}{}^1_\infty(A')\|}\right)^q,$$

Let $A'_1 \subset A'_2$ be bounded open subsets of D'. Extending functions of space $\overset{\circ}{L}{}^1_\infty(A'_1)$ by zero onto the set A'_2, we obtain an inclusion $\overset{\circ}{L}{}^1_\infty(A'_1) \subset \overset{\circ}{L}{}^1_\infty(A'_2)$. Obviously

$$\|f \mid \overset{\circ}{L}{}^1_\infty(A'_1)\| = \|f \mid \overset{\circ}{L}{}^1_\infty(A'_2)\|$$

for every $f \in \overset{\circ}{L}{}^1_\infty(A'_1)$. By the following inequality

$$\begin{aligned}\Phi(A'_1) = \sup_{f\in \overset{\circ}{L}{}^1_\infty(A'_1)} \left(\frac{\|\varphi^* f \mid L^1_q(D)\|}{\|f \mid \overset{\circ}{L}{}^1_\infty(A'_1)\|}\right)^q &= \sup_{f\in \overset{\circ}{L}{}^1_\infty(A'_1)} \left(\frac{\|\varphi^* f \mid L^1_q(D)\|}{\|f \mid \overset{\circ}{L}{}^1_\infty(A'_2)\|}\right)^q \\ &\leq \sup_{f\in \overset{\circ}{L}{}^1_\infty(A'_2)} \left(\frac{\|\varphi^* f \mid L^1_q(D)\|}{\|f \mid \overset{\circ}{L}{}^1_\infty(A'_2)\|}\right)^q = \Phi(A'_2).\end{aligned}$$

the set function Φ is monotone.

Let A'_i, $i \in \mathbb{N}$, be open disjoint subsets at the domain D', $A'_0 = \bigcup\limits_{i=1}^{\infty} A'_i$. Choose arbitrary functions $f_i \in \overset{\circ}{L}{}^1_\infty(A'_i)$ with following properties

$$\|\varphi^* f_i \mid L^1_q(D)\| \geq \big(\Phi(A'_i)\big(1 - \frac{\varepsilon}{2^i}\big)\big)^{\frac{1}{q}} \|f_i \mid \overset{\circ}{L}{}^1_\infty(A'_i)\|$$

and

$$\|f_i \mid \overset{\circ}{L}{}^1_\infty(A'_i)\| = 1,$$

while $i \in \mathbb{N}$. Here $\varepsilon \in (0, 1)$ is a fixed number. Letting $g_N = \sum\limits_{i=1}^{N} f_i$ we obtain

$$\begin{aligned}\|\varphi^* g_N \mid L^1_q(D)\| &\geq \left(\sum_{i=1}^{N} \left(\Phi(A'_i)\left(1 - \frac{\varepsilon}{2^i}\right)\right) \|f_i \mid \overset{\circ}{L}{}^1_\infty(A'_i)\|^q\right)^{1/q} \\ &= \left(\sum_{i=1}^{N} \Phi(A'_i)\left(1 - \frac{\varepsilon}{2^i}\right)\right)^{\frac{1}{q}} \left\|g_N \mid \overset{\circ}{L}{}^1_\infty\Big(\bigcup_{i=1}^{N} A'_i\Big)\right\| \\ &\geq \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon\Phi(A'_0)\right)^{\frac{1}{q}} \left\|g_N \mid \overset{\circ}{L}{}^1_\infty\Big(\bigcup_{i=1}^{N} A'_i\Big)\right\|\end{aligned}$$

since sets, on which the gradients $\nabla\varphi^* f_i$ do not vanish, are disjoint. From the last inequality follows that

$$\Phi(A'_0)^{\frac{1}{q}} \geq \sup \frac{\|\varphi^* g_N \mid L^1_q(D)\|}{\left\|g_N \mid \overset{\circ}{L}{}^1_\infty\Big(\bigcup\limits_{i=1}^{N} A'_i\Big)\right\|} \geq \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon\Phi(A'_0)\right)^{\frac{1}{q}}.$$

Here the upper bound is taken over all above-mentioned functions

$$g_N \in \overset{\circ}{L}{}^1_\infty\Big(\bigcup_{i=1}^{N} A'_i\Big).$$

Since both N and ε are arbitrary, we have finally

$$\sum_{i=1}^{\infty} \Phi(A'_i) \leq \Phi\Big(\bigcup_{i=1}^{\infty} A'_i\Big).$$

The validity of the inverse inequality can be proved in a straightforward manner. Indeed, choose functions $f_i \in \overset{\circ}{L}{}^1_\infty(A'_i)$ such that $\|f_i \mid \overset{\circ}{L}{}^1_\infty(A'_i)\| = 1$.

Letting $g = \sum\limits_{i=1}^{\infty} f_i$ we obtain

$$\|\varphi^* g \mid L^1_q(D)\| \leq \Big(\sum_{i=1}^{\infty} \Phi(A'_i)\|f_i \mid \overset{\circ}{L}{}^1_\infty(A'_i)\|^q\Big)^{1/q} = \Big(\sum_{i=1}^{\infty} \Phi(A'_i)\Big)^{\frac{1}{q}} \Big\|g_N \mid \overset{\circ}{L}{}^1_\infty\Big(\bigcup_{i=1}^{\infty} A'_i\Big)\Big\|,$$

since sets, on which the gradients $\nabla\varphi^* f_i$ do not vanish, are disjoint. From this inequality follows that

$$\Phi\Big(\bigcup_{i=1}^{\infty} A'_i\Big)^{\frac{1}{q}} \leq \sup \frac{\|\varphi^* g \mid L^1_q(D)\|}{\Big\|g \mid \overset{\circ}{L}{}^1_\infty\Big(\bigcup\limits_{i=1}^{\infty} A'_i\Big)\Big\|} \leq \Big(\sum_{i=1}^{\infty} \Phi(A'_i)\Big)^{\frac{1}{q}},$$

where the upper bound is taken over all functions $g \in \overset{\circ}{L}{}^1_\infty\Big(\bigcup\limits_{i=1}^{\infty} A'_i\Big)$.

By the definition of the set function Φ we have

$$\|\varphi^* f \mid L^1_q(D)\|^p \leq \Phi(A')\|f \mid \overset{\circ}{L}{}^1_\infty(A')\|^q$$

Since the support of the function $f \circ \varphi$ is contained in the set $\varphi^{-1}(A')$ we have

$$\int\limits_{\varphi^{-1}(A)} |\nabla(f \circ \varphi)|^q \ dx \leq \Phi(A')\operatorname{esssup}_{y \in A'} |\nabla f|^q(y).$$

Theorem proved.

We recall some basic facts about p-capacity. Let $G \subset \mathbb{R}^n$ be an open set and $E \subset G$ be a compact set. For $1 \leq p \leq \infty$ the p-capacity of the ring (E, G) is defined as

$$\operatorname{cap}_p(E, G) = \inf\Big\{\int_G |\nabla u|^p : u \in L^1_p(G) \cap C_0^\infty(G),\ u \geq 1 \text{ on } E\Big\}.$$

Functions $u \in L^1_p(G) \cap C_0^\infty(G)$, $u \geq 1$ on E, are called admissible functions for ring (E, G).

We need the following estimate of the p-capacity [18].

Lemma 1. Let E be a connected closed subset of an open bounded set $G \subset \mathbb{R}^n, n \geq 2$, and $n - 1 < p < \infty$. Then

$$\operatorname{cap}_p^{n-1}(E, G) \geq c\frac{(\operatorname{diam} E)^p}{|G|^{p-n+1}},$$

where a constant c depends on n and p only.

For readers convenience we will prove this fact.

PROOF. Let d be diameter of set E. Without loss of generality we can suggest, that $d = \operatorname{dist}(0, a)$ for some point $a = (0, .., 0, a_n)$. For arbitrary number t, $0 < t < d$, denote by P_t the hyperplane $x_n = t$.

In the subspace $x_n = 0$ we consider the unit $(n-2)$-dimensional sphere S^{n-2} with the center at the origin and fix an arbitrary point $z \in E \cap P_t$. For every point $y \in S^{n-2}$ denote by $R(y)$ the supremum of numbers r_0 such that $z + ry \in G$ while $0 \le r \le r_0$. Then for every admissible function $f \in C_0^\infty(G)$ the following inequality

$$1 = f(z) - f(z + R(y)y) \le \int_0^{R(y)} |\nabla f(z + ry)|\, dr = \int_0^{R(y)} (|\nabla f(z + ry)| r^{\frac{n-2}{p}}) r^{-\frac{n-2}{p}}\, dr$$

holds. Applying Hölder inequality to the right side of the last inequality, we have

$$1 \le \left(\frac{p-1}{p-n+1}\right)^{p-1} \big(R(y)\big)^{p-n+1} \int_0^{R(y)} |\nabla f(z + ry)|^p r^{n-2}\, dr.$$

Multiplying both sides of this inequality on $((p-1)/(p-n+1))^{1-p} \cdot (R(y))^{n-p-1}$ and integrating by $y \in S^{n-2}$, we obtain

$$\left(\frac{p-1}{p-n+1}\right)^{p-1} \int_{S^{n-2}} \big(R(y)\big)^{p-n+1}\, dy$$

$$\le \int_{S^{n-2}} dy \int_0^{R(y)} |\nabla f(z + ry)|^p r^{n-2}\, dr \le \int_{P_t} |\nabla f|^p\, dz.$$

For the lower estimate of the left integral we use again Hölder inequality. Denote by ω_{n-2} the $n-2$-dimensional area of sphere S^{n-2}. By simple calculations we get

$$\omega_{n-2}^p = \left(\int_{S^{n-2}} dy\right)^p \le \left(\int_{S^{n-2}} \big(R(y)\big)^{n-p-1}\, dy\right)^{n-1} \left(\int_{S^{n-2}} \big(R(y)\big)^{n-1}\, dy\right)^{p+1-n}$$

$$\le ((n-1) m_{n-1}(G \cap P_t))^{p-n+1} \left(\int_{S^{n-2}} \big(R(y)\big)^{n-p-1}\, dy\right)^{n-1}.$$

Here $m_{n-1}(A)$ is $(n-1)$-Lebesgue measure of the set A.

Denote by $u(t) = m_{n-1}(G \cap P_t)$. Using the last estimate we obtain

$$\int_{P_t} |\nabla f|^p\, dz \ge \left(\frac{p-1}{p-n+1}\right)^{1-p} (n-1)^{\frac{n-p-1}{n-1}} \omega_{n-2}^{\frac{p}{n-1}} \big(u(t)\big)^{\frac{n-p-1}{n-1}}.$$

After integrating by $t \in (0, d)$ we have

$$\int_G |\nabla f|^p\, dx \ge \left(\frac{p-1}{p-n+1}\right)^{1-p} (n-1)^{\frac{n-p-1}{n-1}} \omega_{n-2}^{\frac{p}{n-1}} \int_0^d \big(u(t)\big)^{\frac{n-p-1}{n-1}}\, dt.$$

By Hölder inequality

$$d^p = \Big(\int\limits_0^d \, dt\Big)^p \leq \Big(\int\limits_0^d u(t)\, dt\Big)^{p-n+1}\Big(\int\limits_0^d \big(u(t)\big)^{\frac{n-p-1}{n-1}}\, dt\Big)^{n-1}$$

$$\leq |G|^{p-n+1}\Big(\int\limits_0^d \big(u(t)\big)^{\frac{n-p-1}{n-1}}\, dt\Big)^{n-1}.$$

Therefore

$$\int\limits_G |\nabla f|^p\, dx \geq \Big(\frac{p-1}{p-n+1}\Big)^{1-p}(n-1)^{\frac{n-p-1}{n-1}}\omega_{n-2}^{\frac{p}{n-1}}\Big(\frac{d^p}{|G|^{p-n+1}}\Big)^{\frac{1}{n-1}}.$$

Since f is an arbitrary admissible function the required inequality is proved.

Let us define a class BVL of mappings with finite variation. A mapping $\varphi : D \to \mathbb{R}^n$ belongs to the class BVL(D) (i.e., has *finite variation on almost all straight lines*) if it has finite variation on almost all straight lines l parallel to any coordinate axis: for any finite number of points $t_1, ..., t_k$ that belongs to such straight line l

$$\sum_{i=0}^{k-1} |\varphi(t_{i+1}) - \varphi(t_i)| < +\infty.$$

For a mapping φ with finite variation on almost all straight lines, the partial derivatives $\partial\varphi_i/\partial x_j$, $i,j = 1,\dots,n$, exists almost everywhere in D.

Theorem 4. **[11]** Let a homeomorphism $\varphi : D \to D'$ generates a bounded composition operator

$$\varphi^* : L^1_\infty(D') \to L^1_q(D), \ \ q > n-1.$$

Then the inverse homeomorphism $\varphi^{-1} : D' \to D$ belongs to the class BVL(D').

For readers convenience we reproduce here a slightly modified proof of this fact.

Proof. Take an arbitrary n-dimensional open parallelepiped P such that $\overline{P} \subset D'$ and its edges are parallel to coordinate axis. Let us show that φ^{-1} has finite variation on almost all intersection of P and straight lines parallel to x_n-axis.

Let P_0 be the projection of P on the subspace $x_n = 0$, and let I be the projection of P on the coordinate axis x_n. Then $P = P_0 \times I$. The monotone countable-additive function Φ determines a monotone countable additive function of open sets $A \subset P_0$ by the rule $\Phi(A, P_0) = \Phi(A \times I)$. For almost all points $z \in P_0$, the quantity

$$\overline{\Phi'}(z, P_0) = \overline{\lim_{r\to 0}}\Big[\frac{\Phi(B^{n-1}(z,r), P_0)}{r^{n-1}}\Big]$$

is finite [19] (here $B^{n-1}(z,r)$ is the $(n-1)$-dimensional ball of radius $r > 0$ centered at the point z).

The n-dimensional Lebesgue measure $\Psi(U) = |\varphi^{-1}(U)|$, where U is an open sen in D', is a monotone countable additive function and, therefore, also determines a monotone

countable additive function $\Psi(A, P_0) = \Psi(A \times I)$ defined on open sets $A \subset P_0$. Hence $\overline{\Psi'}(z, P_0)$ is finite for almost all points $z \in P_0$.

Choose an arbitrary point $z \in P_0$ where $\overline{\Phi'}(z, P_0) < +\infty$ and $\overline{\Psi'}(z, P_0) < +\infty$. On the section $I_z = \{z\} \times I$ of the parallelepiped P, take arbitrary mutually disjoint closed intervals $\Delta_1, ..., \Delta_k$ with lengths $b_1, ..., b_k$ respectively. Let R_i denote the open set of points for which distances from Δ_i smaller than a given $r > 0$:

$$R_i = \{x \in G : \operatorname{dist}(x, \Delta_i) < r\}.$$

Consider the ring (Δ_i, R_i). Let $r > 0$ be selected so that $r < cb_i$ for $i = 1, \dots, k$, where c is a sufficiently small constant. Then the function $u_i(x) = \operatorname{dist}(x, \Delta_i)/r$ is an admissible for ring (Δ_i, R_i).

By Theorem 3 we have the estimate

$$\|\varphi^* u_i \mid L^1_q(D)\|^q \leq \Phi(A') \|u_i \mid \overset{\circ}{L}{}^1_\infty(A')\|^q$$

for every function u_i, $i = 1, ..., k$.

Hence, for every ring (Δ_i, R_i), $i = 1, ..., k$, the inequality

$$\operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(\Delta_i), \varphi^{-1}(R_i)) \leq \Phi(R_i)^{\frac{1}{q}} \operatorname{cap}_\infty(\Delta_i, R_i)$$

holds.

The function $u_i(x) = \operatorname{dist}(x, \Delta_i)/r$ is admissible for ring (Δ_i, R_i) and we have the upper estimate

$$\operatorname{cap}_\infty(\Delta_i, R_i) \leq |\nabla u_i| = \frac{1}{r}.$$

Applying the lower bound for the capacity of the ring (Lemma 1), we obtain

$$\left(\frac{(\operatorname{diam} \varphi^{-1}(\Delta_i))^{q/(n-1)}}{|\varphi^{-1}(R_i)|^{(q-n+1)/(n-1)}}\right)^{\frac{1}{q}} \leq c_1 \Phi(R_i)^{\frac{1}{q}} \cdot \frac{1}{r}.$$

This inequality gives

$$\operatorname{diam} \varphi^{-1}(\Delta_i) \leq c_2 \left(\frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{\frac{q-n+1}{q}} \cdot \left(\frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$

Summing over $i = 1, \dots, k$ we obtain

$$\sum_{i=1}^k \operatorname{diam} \varphi^{-1}(\Delta_i) \leq c_2 \sum_{i=1}^k \left(\frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{\frac{q-n+1}{q}} \cdot \left(\frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$

Hence

$$\sum_{i=1}^k \operatorname{diam} \varphi^{-1}(\Delta_i) \leq c_2 \left(\sum_{i=1}^k \frac{|\varphi^{-1}(R_i)|}{r^{n-1}}\right)^{\frac{q-n+1}{q}} \cdot \left(\sum_{i=1}^k \frac{\Phi(R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$

Using the Besicovitch type theorem [20] for the estimate of the value of the function Φ in terms of the multiplicity of a cover, we obtain

$$\sum_{i=1}^k \operatorname{diam} \varphi^{-1}(\Delta_i) \leq c_3 \left(\frac{|\varphi^{-1}(\bigcup_{i=1}^k R_i)|}{r^{n-1}}\right)^{\frac{q-n+1}{q}} \cdot \left(\frac{\Phi(\bigcup_{i=1}^k R_i)}{r^{n-1}}\right)^{\frac{n-1}{q}}.$$

Hence

$$\sum_{i=1}^{k} \operatorname{diam} \varphi^{-1}(\Delta_i) \leq c_3 \left(\frac{|\varphi^{-1}(B^{n-1}(z,r), P_0)|}{r^{n-1}} \right)^{\frac{q-n+1}{q}} \cdot \left(\frac{\Phi(B^{n-1}(z,r), P_0)}{r^{n-1}} \right)^{\frac{n-1}{q}}.$$

Because $\overline{\Phi'}(z, P_0) < +\infty$ and $\overline{\Psi'}(z, P_0) < +\infty$ we obtain finally

$$\sum_{i=1}^{k} \operatorname{diam} \varphi^{-1}(\Delta_i) < +\infty.$$

Therefore $\varphi^{-1} \in \mathrm{BVL}(D')$.

Theorem proved.

2. Invertibility of composition operators

Let us recall the change of variable formula for Lebesgue integral [21]. Let a mapping $\varphi : D \to \mathbb{R}^n$ be such that there exists a collection of closed sets $\{A_k\}_1^\infty$, $A_k \subset A_{k+1} \subset D$ for which restrictions $\varphi|_{A_k}$ are Lipschitz mapping on sets A_k and

$$\left| D \setminus \sum_{k=1}^{\infty} A_k \right| = 0.$$

Then there exists a measurable set $S \subset D$, $|S| = 0$ such that the mapping $\varphi : D \setminus S \to \mathbb{R}^n$ has Luzin N-property and the change of variable formula

$$\int_E f \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\mathbb{R}^n \setminus \varphi(S)} f(y) N_f(E, y) \, dy$$

holds for every measurable set $E \subset D$ and every nonnegative Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$. Here $N_f(y, E)$ is the multiplicity function defined as the number of preimages of y under f in E.

If a mapping φ possesses Luzin N-property (the image of a set of measure zero has measure zero), then $|\varphi(S)| = 0$ and the second integral can be rewritten as the integral on $\mathbb{R}^n$. Note, that if a homeomorphism $\varphi : D \to D'$ belongs to the Sobolev space $W^1_{n,\mathrm{loc}}(D)$ then φ has Luzin N-property and the change of variable formula holds [22].

If a mapping $\varphi : D \to \mathbb{R}^n$ belongs to the Sobolev space $W^1_{1,\mathrm{loc}}(D)$ then by [21] there exists a collection of closed sets $\{A_k\}_1^\infty$, $A_k \subset A_{k+1} \subset D$ for which restrictions $f|_{A_k}$ are Lipschitz mapping on sets A_k and

$$\left| D \setminus \sum_{k=1}^{\infty} A_k \right| = 0.$$

Hence for such mappings the previous change of variable formula is correct.

Like in [23] (see also [13]) we define a measurable function

$$\mu(y) = \begin{cases} \left(\frac{|\operatorname{adj} D\varphi|(x)}{|J(x,\varphi)|} \right)_{x=\varphi^{-1}(y)} & \text{if} \quad x \in D \setminus S \quad \text{and} \quad J(x,\varphi) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Because the homeomorphism φ has finite distortion the function $\mu(y)$ is well defined almost everywhere in D'.

The following lemma was proved (but does not formulated) in [13] under an additional assumption that $|D\varphi|$ belongs to the Lorentz space $L^{n-1,n}(D)$.

Lemma 2. Let a homeomorphism $\varphi : D \to D', \varphi(D) = D'$ belongs to the Sobolev space $L^1_q(D)$ for some $q > n-1$. Then the function μ is locally integrable in the domain D'.

Proof. Using the change of variable formula for Lebesgue integral [21] and Luzin N-property of φ we have the following equality

$$\int\limits_{D'} \mu(y)\ dy = \int\limits_{D'\setminus\varphi(S)} \mu(y)\ dy = \int\limits_{D\setminus S} |\mu(\varphi(x))| J(x,\varphi)|\ dx = \int\limits_{D} |\operatorname{adj} D\varphi|(x)\ dx.$$

Applying Hölder inequality, we obtain that for every compact subset $F' \subset D'$

$$\int\limits_{F'} \mu(y)\ dy \leq \int\limits_{F} |\operatorname{adj} D\varphi|(x)\ dx \leq C \int\limits_{F} |D\varphi|^{n-1}(x)\ dx,$$

where $F' = \varphi(F)$. Therefore, μ belongs to $L_{1,\mathrm{loc}}(D')$, since φ belongs to $L^1_q(D)$, $q > n-1$, and as consequence $\varphi \in L^1_{n-1,\mathrm{loc}}(D)$.

Theorem 5. Let a homeomorphism $\varphi : D \to D'$, $\varphi(D) = D'$, has finite distortion, Luzin N-property (the image of a set measure zero is a set measure zero) and generates a bounded composition operator

$$\varphi^* : L^1_\infty(D') \to L^1_q(D), \ \ q > n-1.$$

Then the inverse homeomorphism $\varphi^{-1} : D' \to D$ has integrable first weak derivatives and induces a bounded composition operator

$$(\varphi^{-1})^* : L^1_{q'}(D) \to L^1_1(D'), \quad q' = q/(q-n+1).$$

Proof. We prove that $\varphi^{-1} \in \mathrm{ACL}(D')$. Since absolute continuity is the local property, it is sufficient to prove that the mapping φ^{-1} belongs to ACL on every compact subset of D'. Consider arbitrary cube $Q' \in D'$, $\overline{Q'} \in D$, with edges parallel to coordinate axes, and $Q = \varphi^{-1}(Q')$. For $i = 1, \dots n$ we will use a notation: $Y_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$,

$$F_i(x) = (\varphi_1(x), \dots, \varphi_{i-1}(x), \varphi_{i+1}(x), \dots, \varphi_n(x))$$

and Q'_i is the intersection of the cube Q' with a line $Y_i = \mathrm{const}$.

Using the change of variable formula and the Fubini theorem [24] we obtain the following estimate

$$\int\limits_{F_i(Q)} H^{n-1}(dY_i) \int\limits_{Q'_i} \mu(y)\ H^1(dy) = \int\limits_{Q'} \mu(y)\ dy = \int\limits_{Q} |\operatorname{adj} D\varphi|(x)\ dx < +\infty.$$

Hence for almost all $Y_i \in F_i(Q)$

$$\int\limits_{Q_i'} \mu(y)\ H^1(dy) < +\infty.$$

Let $\mathrm{ap}\, J\varphi(x)$ be an approximate Jacobian of the trace of the mapping φ on the set $\varphi^{-1}(Q_i')$ [24]. Consider a point $x \in Q$ in which there exists a non-generated approximate differential $\mathrm{ap}\, Df(x)$ of the mapping $\varphi : D \to D'$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping induced by this approximate differential $\mathrm{ap}\, Df(x)$. We denote by the symbol P the image of the unit cube Q_0 under the linear mapping L and by P_i the intersection of P with the image of the line $x_i = 0$. Let d_i be a length of P_i. Then

$$d_i \cdot |\operatorname{adj} DF_i|(x) = |Q_0| = |J(x,\varphi)|.$$

So, since $d_i = \mathrm{ap}\, J\varphi(x)$ we obtain that for almost all $x \in Q \setminus Z$, $Z = \{x \in D : J(x,\varphi) = 0\}$, we have

$$\mathrm{ap}\, J\varphi(x) = \frac{|J(x,\varphi)|}{|\operatorname{adj} DF_i|(x)}.$$

So, we have for arbitrary compact set $A' \subset Q_i'$, and for almost all $Y_i \subset F_i(Q)$, the following inequality:

$$H^1(\varphi^{-1}(A')) \leq \int\limits_{\varphi^{-1}(A')} \frac{|\operatorname{adj} D\varphi|(x)}{|\operatorname{adj} DF_i|(x)}\ H^1(dx)$$
$$= \int\limits_{\varphi^{-1}(A')} \frac{|\operatorname{adj} D\varphi|(x)}{|J(x,\varphi)|} \cdot \frac{|J(x,\varphi)|}{|\operatorname{adj} DF_i|(x)}\ H^1(dx) = \int\limits_{\varphi^{-1}(A')} \mu(\varphi(x))\, \mathrm{ap}\, J\varphi(x)\ H^1(dx).$$

By using the change of variable formula for the Lebesgue integral [24, 25] we obtain

$$H^1(f^{-1}(A')) \leq \int\limits_{A'} \mu(y)\ H^1(dy) < +\infty.$$

Therefore, the mapping φ^{-1} is absolutely continuous on almost all lines in $D'$and is a weakly differentiable mapping.

Since the homeomorphism φ has Luzin N-property then preimage of a set positive measure is a set positive measure. Hence, the volume derivative of the inverse mapping

$$J_{\varphi^{-1}}(y) = \lim_{r\to 0} \frac{|\varphi^{-1}(B(y,r))|}{|B(y,r)|} > 0$$

almost everywhere in D'. So $J(y,\varphi^{-1}) \neq 0$ for almost all points $y \in D$. Integrability of the q'-distortion follows from the inequality

$$|D\varphi^{-1}|(y) \leq |D\varphi(x)|^{n-1}/|J(x,\varphi)|$$

which holds for almost all points $y = \varphi(x) \in D'$.

Indeed, with the help of the change of variable formula, we have

$$\int\limits_{D'} \left(\frac{|D\varphi^{-1}(y)|^{q'}}{|J(y,\varphi^{-1})|}\right)^{\frac{1}{q'-1}} dy = \int\limits_{D'} \left(\frac{|D\varphi^{-1}(y)|}{|J(y,\varphi^{-1})|}\right)^{\frac{q'}{q'-1}} |J(y,\varphi^{-1})|\, dy$$

$$\leq \int\limits_{D} \left(\frac{|D\varphi^{-1}(\varphi(x))|}{|J(\varphi(x),\varphi^{-1})|}\right)^{\frac{q'}{q'-1}} dx \leq \int\limits_{D} |D\varphi(x)|^q\, dx < +\infty,$$

since by Theorem 2 φ belongs to $L^1_q(D)$.

The boundedness of the composition operator follows from integrability of the p'-distortion [2]. The theorem proved.

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